

## SOME DIVISIBILITY PROPERTIES OF BINOMIAL COEFFICIENTS

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ABSTRACT. In this paper, we aim to give full proofs or partial answers for the following three conjectures of V. J. W. Guo and C. Krattenthaler: (1) Let  $a > b$  be positive integers,  $\alpha, \beta$  be any integers and  $p$  be a prime satisfying  $\gcd(p, a) = 1$ . Then there exist infinitely many positive integers  $n$  for which  $\binom{an+\alpha}{bn+\beta} \equiv r \pmod{p}$  for all integers  $r$ ; (2) For any odd prime  $p$ , there are no positive integers  $a > b$  such that  $\binom{an}{bn} \equiv 0 \pmod{pn-1}$  for all  $n \geq 1$ ; (3) For any positive integer  $m$ , there exist positive integers  $a$  and  $b$  such that  $am > b$  and  $\binom{amn}{bn} \equiv 0 \pmod{an-1}$  for all  $n \geq 1$ . Moreover, we show that for any positive integer  $m$ , there are positive integers  $a$  and  $b$  such that  $\binom{amn}{bn} \equiv 0 \pmod{an-a}$  for all  $n \geq 1$ .

## 1. INTRODUCTION

Binomial coefficients constitute an important class of numbers that arise naturally in mathematics, namely as coefficients in the expansion of the polynomial  $(x+y)^n$ . Accordingly, they appear in various mathematical areas. An elementary property of binomial coefficients is that  $\binom{n}{m}$  is divisible by a prime  $p$  for all  $1 < m < n$  if and only if  $n$  is a power of  $p$ . A much more technical result is due to Lucas, which asserts that

$$\binom{n}{m} \equiv \binom{n_0}{m_0} \binom{n_1}{m_1} \cdots \binom{n_k}{m_k} \pmod{p},$$

in which  $n = n_0 + n_1p + \cdots + n_kp^k$  and  $m = m_0 + m_1p + \cdots + m_kp^k$  the  $p$ -adic expansions of the non-negative integers  $n$  and  $m$ , respectively. We note that  $0 \leq m_i, n_i < p$ , for all

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$i = 0, \dots, k$ . In 1819, Babbage [1] revealed the following congruences for all odd prime  $p$ :

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^2}.$$

In 1862, *Wolstenholme* [7] strengthened the identity of Babbage by showing that the same congruence holds modulo  $p^3$  for all prime  $p \geq 5$ . This identity was further generalized by Ljunggren in 1952 to  $\binom{np}{mp} \equiv \binom{n}{m} \pmod{p^3}$  and even more to  $\binom{np}{mp} / \binom{n}{m} \equiv 1 \pmod{p^q}$  by Jacobsthal for all positive integers  $n > m$  and primes  $p \geq 5$ , in which  $p^q$  is any power of  $p$  dividing  $p^3 mn(n-m)$ . Note that the number  $q$  can be replaced by a large number if  $p$  divides  $B_{p-1}$ , the  $(p-3)$ 'th Bernoulli number. Arithmetic properties of binomial coefficients are studied extensively in the literature and we may refer the interested reader to [?] for an account of Wolstenholme's theorem. Recently, *Guo* and *Krattenthaler* [2] studied a similar problem and proved the following conjecture of *Sun* [5].

**Theorem 1.1.** *Let  $a$  and  $b$  be positive integers. If  $bn+1$  divides  $\binom{an+bn}{an}$  for all sufficiently large positive integers  $n$ , then each prime factor of  $a$  divides  $b$ . In other words, if  $a$  has a prime factor not dividing  $b$ , then there are infinitely many positive integers  $n$  for which  $bn+1$  does not divide  $\binom{an+bn}{an}$ .*

They also stated several conjectures among which are the following, which we aim to give full proofs for two conjectures and a partial answer for one of them.

**Conjecture 1.2** ([2, Conjecture 7.1]). Let  $a > b$  be positive integers,  $\alpha, \beta$  be any integers and  $p$  be a prime satisfying  $\gcd(p, a) = 1$ . Then there exist infinitely many positive integers  $n$  for which

$$\binom{an + \alpha}{bn + \beta} \equiv r \pmod{p}$$

for all integers  $r$ .

**Conjecture 1.3** ([2, Conjecture 7.2]). For any odd prime  $p$ , there are no positive integers  $a > b$  such that

$$\binom{an}{bn} \equiv 0 \pmod{pn-1}$$

for all  $n \geq 1$ .

**Conjecture 1.4** ([2, Conjecture 7.3]). For any positive integer  $m$ , there exist positive integers  $a$  and  $b$  such that  $am > b$  and

$$\binom{amn}{bn} \equiv 0 \pmod{an-1}$$

for all  $n \geq 1$ .

Moreover, we show that for any positive integer  $m$ , there are positive integers  $a$  and  $b$  such that  $\binom{amn}{bn} \equiv 0 \pmod{an - a}$  for all  $n \geq 1$ .

## 2. CONJECTURE 1.3

Our first result is a more precise version of Conjecture 1.3 in this case that  $a \not\equiv 0 \pmod{p}$  and we obtain some divisibility property of binomial coefficients.

**Theorem 2.1.** *For any odd prime  $p$ , there are no positive integers  $a > b$  with  $a \not\equiv 0 \pmod{p}$  such that*

$$\binom{an}{bn} \equiv 0 \pmod{pn - 1},$$

for all  $n \geq 1$ .

*Proof.* There are two cases.

**Case I.**  $a \not\equiv 0 \pmod{p}$  and  $b \not\equiv 0 \pmod{p}$ .

There is an  $1 \leq s \leq p - 1$  such that  $sb \equiv 1 \pmod{p}$ . We may write

$$\begin{aligned} sa &= pQ + r, \quad 1 \leq r \leq p - 1 \\ sb &= pQ' + 1. \end{aligned}$$

Choose  $t$  such that  $st \equiv -1 \pmod{p}$ . Thus there is a  $k$  with  $st = kp - 1$ . We claim that

$$(pn + t)(p + s) = pK - 1 \mid \binom{aK}{bK},$$

where  $K = pn + ns + t + k$ . By Dirichlet's theorem, there are infinitely many primes of the form  $pn + t$ . If  $pn + t$  is prime, Lucas' theorem implies that

$$\begin{aligned} \binom{aK}{bK} &= \binom{a(pn + ns + t + k)}{b(pn + ns + t + k)} \\ &= \binom{a(pn + t) + a(ns + k)}{b(pn + t) + b(ns + k)} \\ &= \binom{a(pn + t) + Q(pn + t) + rn + ak - Qt}{b(pn + t) + Q'(pn + t) + n + bk - Q't} \\ &\equiv \binom{a + Q}{b + Q'} \binom{rn + ak - Qt}{n + bk - Q't} \pmod{pn + t}, \end{aligned}$$

since for sufficiently large  $n$  we have  $rn + ak - Qt, n + bk - Q't < pn + t$ .

Now we have

$$\begin{aligned}
s(n + bk - Q't) &= sn + (pQ' + 1)k - Q'(pk - 1) \\
&= sn + k + Q' \\
&\leq srn + rk + Q \\
&\leq srn + (pQ + r)k - Q(pk - 1) = s(rn + ak - Qt).
\end{aligned}$$

Whence  $\binom{rn+ak-Qt}{n+bk-Q't} \neq 0$ .

**Case II.**  $a \not\equiv 0 \pmod{p}$  and  $b \equiv 0 \pmod{p}$ .

We should have

$$0 \equiv \binom{aK}{bK} \equiv \binom{a+Q}{b+Q'} \binom{rn+ak-Qt}{bk-Q't} \pmod{pn+t},$$

where  $K = pn + t + k$ . That is impossible for sufficiently large  $n$ .  $\square$

In the next theorem we aim at considering the case  $a = cp$  and  $b = pk + r$ , where  $1 \leq r \leq p - 1$  and we give a partial answer for Conjecture 1.3 in this case.

We know that for each prime number  $p$  and  $\varepsilon > 0$  there is a real number  $M_p(\varepsilon)$  such that for each  $x \geq M_p(\varepsilon)$  there is a prime number  $q$  in the interval  $(x, (1 + \varepsilon)x)$  with  $q \equiv -1 \pmod{p}$  [3]. Moreover, there is a real number  $M'_p(\varepsilon)$  such that for each  $x \geq M'_p(\varepsilon)$  there are at least two prime numbers  $q, q'$  in the interval  $(x, (1 + \varepsilon)x)$  with  $q, q' \equiv -1 \pmod{p}$ .

In the following we may assume  $b < c(p - r)$ , since if  $b \geq c(p - r)$  then  $\binom{pcn}{bn} = \binom{pcn}{b'n}$ , where  $b' = pc - b$ . We have  $b' = pk' + r'$ , where  $k' = c - k - 1$ ,  $r' = p - r$  and  $b' < c(p - r')$ .

**Theorem 2.2.** *Let  $p$  be an odd prime,  $1 \leq r \leq p - 1$  and  $\gamma = \frac{p^2r+p^2+r^2-pr}{p^2(p+1)}$ .*

- i. *If  $pk + r \leq cp(1 - \gamma)$  then there are no positive integers  $c \geq M_p(\frac{(p-r)^2}{pr(p+1)})$  and  $k$  such that*

$$\binom{pcn}{(pk+r)n} \equiv 0 \pmod{pn-1}.$$

*for all  $n \geq 1$ .*

- ii. *If  $cp(1 - \gamma) < pk + r < c(p - r)$  and  $r \leq \frac{p-3}{2}$  then there are no positive integers  $k \geq 2M'_p(\frac{p-(2r+1)}{p+1})$  and  $c$  such that*

$$\binom{pcn}{(pk+r)n} \equiv 0 \pmod{pn-1}.$$

*for all  $n \geq 1$ .*

*Proof.* i. Put  $b = pk + r$ . We have  $-\frac{b}{c} \geq (\gamma - 1)p$ . Thus

$$\frac{\frac{p(c-k)}{r} - 1}{c} = \frac{p(c-k) - r}{rc} = \frac{p}{r} - \frac{b}{rc} \geq \frac{p}{r} + \frac{\gamma p}{r} - \frac{p}{r} = 1 + \frac{(p-r)^2}{pr(p+1)} > 1.$$

Now since  $c \geq M_p(\frac{(p-r)^2}{pr(p+1)})$ , there is a prime number  $pn - 1$  with

$$c < pn - 1 < \frac{p(c-k)}{r} - 1.$$

This implies the result, since  $k \leq rn + k \leq c < pn - 1$  and Lucas' theorem implies

$$\binom{pcn}{(pk+r)n} = \binom{c(pn-1)+c}{k(pn-1)+rn+k} \equiv \binom{c}{k} \binom{c}{rn+k} \pmod{pn-1}.$$

ii. For  $\alpha = \frac{p+1}{2(p-r)}$  we have

$$\frac{k}{\alpha k} = \frac{2(p-r)}{p+1} = 1 + \frac{p-(2r+1)}{p+1} > 1$$

and since  $\alpha k \geq M'_p(\frac{p-(2r+1)}{p+1})$ , there are two prime numbers  $pm - 1, pn - 1$  with  $\alpha k < pm - 1 < pn - 1 < k$ . We have

$$k = \frac{b-r}{p} < \frac{c(p-r)-r}{p} = c - \frac{r(c+1)}{p} < c.$$

Furthermore,

$$rn + k < r \cdot \frac{k+1}{p} + k = \frac{rk+b}{p} < \frac{rk+c(p-r)}{p} < \frac{rc+c(p-r)}{p} = c.$$

Moreover,

$$\frac{c}{k} < \frac{b}{k(1-\gamma)p} = \frac{kp+r}{kp \cdot \frac{(p^2+r)(p-r)}{p^2(p+1)}} \leq \frac{p+1}{p-r},$$

where the last inequality is true since  $k \geq p$ . We can therefore deduce that

$$pn - 1 < k < c < 2 \cdot \frac{p+1}{2} k = 2\alpha k < 2(pm - 1).$$

We have  $c+1 \leq 2(pm-1) < 2(pn-1)$ . Write  $c = (pn-1) + R$  and  $rn+k = (pn-1) + R'$ .

We know that  $pn - 1 > R > R'$ . Now Lucas' theorem implies

$$\binom{pcn}{(pk+r)n} = \binom{c(pn-1)+(pn-1)+R}{k(pn-1)+(pn-1)+R'} \equiv \binom{c+1}{k+1} \binom{R}{R'} \pmod{pn-1}.$$

The latter is not congruent to 0, since

$$\lfloor \frac{c+1}{pn-1} \rfloor - \lfloor \frac{k+1}{pn-1} \rfloor - \lfloor \frac{c-k}{pn-1} \rfloor \leq 1 - 1 - 0 = 0.$$

□

**Lemma 2.3.** *Let  $p$  be an odd prime,  $1 \leq r \leq p-2$ ,  $j = \lfloor \frac{p}{p-r} \rfloor$  and  $\alpha = \frac{p}{(p-r)(j+1)}$ . Then there is an  $0 < \varepsilon(p, r) < 1$  such that*

$$\alpha < \frac{p + \varepsilon(p, r)}{(p-r)(j+1)} < \frac{\frac{r}{p-r}}{j-1 + \frac{r}{p}}.$$

*Proof.* A simple verification shows that

$$\alpha < \frac{\frac{r}{p-r}}{j-1 + \frac{r}{p}}$$

if and only if  $p-r \nmid p$  or equivalently  $r \neq p-1$ . This implies the existence of  $\varepsilon(p, r)$ .  $\square$

On the other hand, let  $c = j(pn-1) + R$ ,  $0 \leq R \leq pn-2$  and  $rn+k = (pn-1) + R'$ ,  $0 \leq R' \leq pn-2$  and suppose  $pn-1 = \theta k$ , where  $\alpha < \theta < \beta$ . Then by Lemma 2.3,

$$\begin{aligned} R' + j\theta k &= k - (pn-1) + rn + j\theta k \\ &= k - \theta k + r \cdot \frac{\theta k + 1}{p} + j\theta k \\ &= k(1 + (-1 + \frac{r}{p} + j)\theta) + \frac{r}{p} \\ &\leq k(1 + (-1 + \frac{r}{p} + j)\beta) + \frac{r}{p} \\ &= k(1 + (j-1 + \frac{r}{p})\beta) + \frac{r}{p} \\ &< k(1 + \frac{r}{p-r}) + \frac{r}{p} \\ &< \frac{p}{p-r}k + \frac{r}{p-r} \\ &< c. \end{aligned}$$

Hence

$$R = c - j(pn-1) = c - j\theta k > R'.$$

This shows that

$$\lfloor \frac{c}{pn-1} \rfloor - \lfloor \frac{rn+k}{pn-1} \rfloor - \lfloor \frac{c - (rn+k)}{pn-1} \rfloor = j-1 - (j-1 + \lfloor \frac{R-R'}{pn-1} \rfloor) = 0.$$

### 3. CONJECTURE 1.4

In this section, using only properties of the  $p$ -adic valuation we give an inductive proof of Conjecture 7.3 of [2]. For  $n \in \mathbb{N}$  and a prime  $p$ , the  $p$ -adic valuation of  $n$ , denoted by  $\nu_p(n)$  is the highest power of  $p$  that divides  $n$ . The expansion of  $n \in \mathbb{N}$  in base  $p$  is

written as  $n = n_0 + n_1p + \dots + n_kp^k$  with integers  $0 \leq n_i \leq p-1$  and  $n_k \neq 0$ . Legendre's classical formula for the factorials  $\nu_p(n!) = \sum_{i=1}^{\infty} \lfloor \frac{n}{p^i} \rfloor$  appears in elementary textbooks.

**Theorem 3.1.** *For any positive integer  $m$ , there are positive integers  $a$  and  $b$  such that  $am > b$  and*

$$\binom{amn}{bn} \equiv 0 \pmod{an-1}$$

for all  $n \geq 1$ .

*Proof.* Let  $p_1 = 2 < p_2 = 3 < p_3 = 5 < \dots$  be the sequence of prime numbers. Choose  $t$  such that  $p_t > 3m$  and put

$$\begin{aligned} a &= 6p_3 \dots p_t, \\ b &= 4p_3 \dots p_t. \end{aligned}$$

Let  $n$  be a positive integer and  $q^\alpha \mid an-1$  for some prime number  $q$ . We aim at showing that  $q^\alpha \mid \binom{amn}{bn}$ . This of course proves that  $an-1 \mid \binom{amn}{bn}$ .

Write  $bn$  in base  $q$  in the form  $\sum_{j=0}^N r_j q^j$ , where  $N = \alpha-1$  or  $\alpha$  since  $bn \geq q^{\alpha-1}$ . At first we show that  $m < r_0$ . We have

$$r_0 \equiv bn = 4p_3 \dots p_t n \equiv 2 \cdot 3^* \cdot 6p_3 \dots p_t n = 2 \cdot 3^* an \equiv 2 \cdot 3^* \pmod{q},$$

where  $3^*$  is the inverse of 3 mod  $q$ . We know that

$$3^* = \begin{cases} \frac{q+1}{3} & \text{if } 3 \mid q+1 \\ \frac{2q+1}{3} & \text{if } 3 \mid q+2 \end{cases}$$

Note that  $3^*$  exists since  $q \neq 3$ . We thus have

$$r_0 = 2 \cdot 3^* = \begin{cases} 2 \cdot \frac{q+1}{3} > \frac{q}{3} & \text{if } 3 \mid q+1 \\ 2 \cdot \frac{2q+1}{3} - q = \frac{q+2}{3} > \frac{q}{3} & \text{if } 3 \mid q+2 \end{cases}$$

We have  $\gcd(q, p_1 p_2 \dots p_t) = 1$ . Hence  $q > p_t > 3m$ . This shows that  $m < r_0$ . We therefore have

$$\lfloor \frac{bn-m}{q^i} \rfloor = \lfloor \frac{\sum_{j=0}^N r_j q^j - m}{q^i} \rfloor = \lfloor \sum_{j=i}^N r_j q^{j-i} + \frac{\sum_{j=1}^{i-1} r_j q^j + r_0 - m}{q^i} \rfloor = \sum_{j=i}^N r_j q^{j-i} = \lfloor \frac{bn}{q^i} \rfloor.$$

Now let  $an - 1 = kq^\alpha$ , where  $\gcd(k, q) = 1$ . We evaluate the  $q$ -adic valuation  $v_q\left(\binom{amn}{bn}\right)$ . If  $N = \alpha$  then

$$\begin{aligned}
v_q\left(\binom{amn}{bn}\right) &\geq \sum_{i=1}^{\alpha} \left( \left\lfloor \frac{amn}{q^i} \right\rfloor - \left\lfloor \frac{bn}{q^i} \right\rfloor - \left\lfloor \frac{(am-b)n}{q^i} \right\rfloor \right) \\
&\geq \sum_{i=1}^{\alpha} \left( mkq^{\alpha-i} - \left\lfloor \frac{bn}{q^i} \right\rfloor - \left\lfloor \frac{(am-b)n}{q^i} \right\rfloor \right) \\
&= \sum_{i=1}^{\alpha} \left( mkq^{\alpha-i} - \left\lfloor \frac{bn}{q^i} \right\rfloor - \left\lfloor \frac{mkq^\alpha + m - bn}{q^i} \right\rfloor \right) \\
&= \sum_{i=1}^{\alpha} \left( mkq^{\alpha-i} - \left\lfloor \frac{bn}{q^i} \right\rfloor - mkq^{\alpha-i} - \left\lfloor \frac{m - bn}{q^i} \right\rfloor \right) \\
&= \sum_{i=1}^{\alpha} \left( - \left\lfloor \frac{bn}{q^i} \right\rfloor - \left\lfloor -\frac{bn - m}{q^i} \right\rfloor \right) \\
&= \sum_{i=1}^{\alpha} \left( - \left\lfloor \frac{bn}{q^i} \right\rfloor + \left\lfloor \frac{bn - m}{q^i} \right\rfloor + 1 \right) \\
&= \sum_{i=1}^{\alpha} 1 = \alpha,
\end{aligned}$$

since  $\frac{bn-m}{q^i}$  is not an integer.

On the other hand, if  $N = \alpha - 1$  then

$$\begin{aligned}
v_q\left(\binom{amn}{bn}\right) &= mk + \sum_{i=1}^{\alpha-1} \left( \left\lfloor \frac{amn}{q^i} \right\rfloor - \left\lfloor \frac{bn}{q^i} \right\rfloor - \left\lfloor \frac{(am-b)n}{q^i} \right\rfloor \right) \\
&\geq mk + \alpha - 1 \geq \alpha.
\end{aligned}$$

Thus  $q^\alpha \mid \binom{amn}{bn}$ .

□

**Theorem 3.2.** *For any positive integer  $m$ , there are positive integers  $a$  and  $b$  such that*

$$\binom{amn}{bn} \equiv 0 \pmod{an - a}$$

for all  $n \geq 1$ .

*Proof.* Let  $p_1 = 2 < p_2 = 3 < p_3 = 5 < \dots$  be the sequence of prime numbers. For a positive integer  $t$  put

$$b = ap_3 \dots p_t.$$



Let  $n$  be a positive integer and  $q^\alpha \mid an - a$  for some prime number  $q$ . It is sufficient  $q^\alpha \mid \binom{amn}{bn}$ , this concludes the proof. Let  $b = r_0 + r_1q + \dots + r_\alpha q^\alpha$  is the  $q$ -adic expansions of  $b$  where  $0 \leq r_i \leq q - 1$ . We have  $\gcd(q, p_1 p_2 \dots p_t) = 1$  and  $r_0 \equiv b \pmod{q}$ . Now, let  $an = a + kq^\alpha$  where  $\gcd(k, q) = 1$ . We evaluate the  $q$ -adic valuation  $v_q(\binom{amn}{bn})$ . We have

$$\begin{aligned} v_q\left(\binom{amn}{bn}\right) &= v_q\left(\frac{(amn)!}{(bn)!(amn - bn)!}\right) \\ &= \sum_{i=1}^{\alpha} \left( \left\lfloor \frac{amn}{q^i} \right\rfloor - \left\lfloor \frac{bn}{q^i} \right\rfloor - \left\lfloor \frac{anm - bn}{q^i} \right\rfloor \right) \\ &= \sum_{i=1}^{\alpha} \left( \left\lfloor \frac{mkq^\alpha + ma}{q^i} \right\rfloor - \left\lfloor \frac{anp_1 p_2 \dots p_t}{q^i} \right\rfloor - \left\lfloor \frac{anm - anp_1 p_2 \dots p_t}{q^i} \right\rfloor \right) \\ &= \sum_{i=1}^{\alpha} \left( \left\lfloor \frac{am}{q^i} \right\rfloor - \left\lfloor \frac{b}{q^i} \right\rfloor - \left\lfloor \frac{am - b}{q^i} \right\rfloor \right) \end{aligned}$$

So, it is sufficient to show for any  $m \in \mathbb{Z}^+$  the inequality

$$\left\lfloor \frac{am}{q^i} \right\rfloor - \left\lfloor \frac{b}{q^i} \right\rfloor - \left\lfloor \frac{am - b}{q^i} \right\rfloor \geq 0 \quad (3.1)$$

Let  $\frac{ma}{q^\alpha} = s + r$  and  $\frac{b}{q^\alpha} = s' + r'$  where  $s, s' \in \mathbb{Z}^+$  and  $0 \leq r, r' \leq 1$ , then  $\left\lfloor \frac{am}{q^i} \right\rfloor - \left\lfloor \frac{b}{q^i} \right\rfloor = s + s'$  and  $\left\lfloor \frac{am - b}{q^i} \right\rfloor = s + s'$  or  $s + s' - 1$ . Therefore 3.1 holds and this concludes the proof.  $\square$

#### 4. CONJECTURE 1.2

Maxim Vsemirnov [6] proved that the conjecture 1.2 is not true for  $p = 5$ . He also proved the following theorem:

**Theorem 4.1.** *Let  $p = 5, a = 4, b = 2$ . If  $(\alpha, \beta) \in \{(0, 0), (1, 0), (1, 1)\}$  then*

$$\binom{4n + \alpha}{2n + \beta} \equiv 0, 1 \text{ or } 4 \pmod{5};$$

*If  $(\alpha, \beta) \in \{(2, 1), (3, 1), (3, 2)\}$  then*

$$\binom{4n + \alpha}{2n + \beta} \equiv 0, 2 \text{ or } 3 \pmod{5};$$

In the following, we give a proof for a special case of Conjecture 1.2 We know that if  $\gcd(x, y) = 1$  then there is an integer  $1 \leq x' \leq y - 1$  such that  $y \mid xx' - 1$ . We denote this  $x'$  by  $\text{Inv}_y(x)$ . Moreover, for an integer  $x$  we denote the  $p$ -adic valuation of  $x$  by  $v_p(x)$ .

**Theorem 4.2.** *Let  $a$  and  $b$  be positive integers with  $a > b$ , let  $\alpha$  and  $\beta$  be integers and let  $d = \gcd(a, b)$ ,  $c = \frac{a}{d}$ ,  $e = \gcd(p-1, a)$ . Furthermore, let  $p$  be a prime such that  $p > a + 2b$ . Then*

- i. *if  $e < c$  or  $v_2(a) \leq v_2(p-1)$  then for each  $r = 0, 1, \dots, p-1$ , there are infinitely many positive integers  $n$  such that*

$$\begin{pmatrix} an + \alpha \\ bn + \beta \end{pmatrix} \equiv r \pmod{p};$$

- ii. *if  $e \geq c$  and  $v_2(a) > v_2(p-1)$  then for each*

$$r \notin \{(2\mu - 1)e + p + \alpha - 2 + r' : 0 \leq r' \leq e - c, \lceil \frac{e + 2 - p - \alpha}{2e} \rceil \leq \mu \leq \lceil \frac{c + 1 - \alpha}{2e} \rceil\},$$

*there are infinitely many positive integers  $n$  such that*

$$\begin{pmatrix} an + \alpha \\ bn + \beta \end{pmatrix} \equiv r \pmod{p}.$$

*Proof.* By Euler's totient theorem, we have  $p^{\varphi(a)} \equiv 1 \pmod{a}$ , since  $\gcd(p, a) = 1$ . For an arbitrary positive integer  $N$ , put  $u = N\varphi(a)$ . Thus

$$p^{ui} \equiv 1 \pmod{a}, \quad i \in \mathbb{N}.$$

In particular, there is an  $m$  such that  $p^u - 1 = am$ . Thus  $m = -\text{Inv}_p(a)$ . Put  $t = (p-1) - r$ . Write  $c - t - \alpha = \mu e + \rho$ , where  $0 \leq \rho \leq e - 1$ . Note that  $e \mid c - t - \alpha - \rho$ . Suppose

$$\varepsilon = \begin{cases} 0 & \text{if } \rho \leq c - 2 \\ 1 & \text{otherwise} \end{cases}$$

If  $e < c$  then put

$$\begin{aligned} K &= \frac{c - t - \alpha - \rho}{e} \cdot (p\text{Inv}_{\frac{a}{e}}(\frac{p(p-1)}{e})) \\ &\quad + (c - t - \alpha - \rho)am\text{Inv}_p(a+1) - (\beta - 1)a^2m\text{Inv}_p(b(a+1)) + Lpa, \end{aligned}$$

where  $L$  is sufficiently large so that  $K > 1$ . Note that  $\varepsilon = 0$  in this case, since  $\rho \leq e - 1 \leq c - 2$ .

If  $e \geq c$  and  $v_2(a) \leq v_2(p-1)$  then put

$$\begin{aligned} K &= \frac{c - t - \alpha - \rho}{e} \cdot (p\text{Inv}_{\frac{a}{e}}(\frac{p(p-1)(1+\varepsilon)}{e})) \\ &\quad + (c - t - \alpha - \rho)am\text{Inv}_p(a+1) - (\beta - 1)a^2m\text{Inv}_p(b(a+1)) + Lpa, \end{aligned}$$

where  $L$  is sufficiently large so that  $K > 1$ . Note that  $\text{Inv}_{\frac{a}{e}}(1 + \varepsilon)$  exists, since  $\frac{a}{e}$  is odd in this case.

Finally, if  $e \geq c$  and  $v_2(a) > v_2(p-1)$  then put

$$\begin{aligned} K &= \frac{c-t-\alpha-\rho}{(1+\varepsilon)e} \cdot \left( p\text{Inv}_{\frac{a}{(1+\varepsilon)e}}\left(\frac{p(p-1)}{e}\right) \right) \\ &\quad + (c-t-\alpha-\rho)am\text{Inv}_p(a+1) - (\beta-1)a^2m\text{Inv}_p(b(a+1)) + Lpa, \end{aligned}$$

where  $L$  is sufficiently large so that  $K > 1$ . Note that  $\frac{c-t-\alpha-\rho}{e}$  is even by our assumption on  $r$  in this case.

In each of the above cases we have

$$\begin{aligned} K(p-1)(1+\varepsilon) &\equiv c-t-\alpha-\rho \pmod{a}, \\ mb(K(p-1)(a+1+\varepsilon) - (c-t-\alpha-\rho)) &\equiv \beta-1 \pmod{p}. \end{aligned}$$

Now choose

$$M = K(p-1)(d(c-1)+1) - (c-1) + \rho,$$

and

$$\begin{aligned} \mathbb{I}_2 &= \{M - k(c-1) : k = 0, 1, \dots, K(p-1)(d+\varepsilon) - 1\}, \\ \mathbb{I}_1 &= \{1, 2, \dots, M\} \setminus \mathbb{I}_2. \end{aligned}$$

We have

$$\begin{aligned} &p^{u(M+1)} - t - \sum_{i \in \mathbb{I}_1} p^{ui} - \sum_{i \in \mathbb{I}_2} 2p^{ui} - \alpha \\ &\equiv 1 - t - (M - K(p-1)(d+\varepsilon)) - 2K(p-1)(d+\varepsilon) - \alpha \\ &= 1 - t - \alpha - K(p-1)(a+1+\varepsilon) + (c-1) - \rho \\ &\equiv c - t - \alpha - \rho - K(p-1)(1+\varepsilon) \\ &\equiv 0 \pmod{a}. \end{aligned}$$

Hence, there is a positive integer  $n$  such that

$$an + \alpha = p^{u(M+1)} - t - \sum_{i \in \mathbb{I}_1} p^{ui} - \sum_{i \in \mathbb{I}_2} 2p^{ui}.$$

Write  $an + \alpha$  in base  $p$  as the form  $\sum_{s=0}^{u(M+1)} a_s p^s$ . Then we have

$$a_s = \begin{cases} p-1-t & \text{if } s=0 \\ p-2 & \text{if } s=ui \text{ for some } i \in \mathbb{I}_1 \\ p-3 & \text{if } s=ui \text{ for some } i \in \mathbb{I}_2 \\ p-1 & \text{otherwise} \end{cases}$$

We now aim at finding digits of  $bn + \beta$  in base  $p$ . If  $bn + \beta = \sum_{s=0}^{u(M+1)} b_s p^s$  then  $b_s$  is the remainder of  $\lfloor \frac{bn+\beta}{p^s} \rfloor \bmod p$ . In fact, we need to find  $b_s$  for  $s = 0, u, 2u, \dots, Mu$ .

We now have

$$\begin{aligned}
bn + \beta &= \frac{b}{a} \left( p^{u(M+1)} - t - \sum_{i \in \mathbb{I}_1} p^{ui} - \sum_{i \in \mathbb{I}_2} 2p^{ui} \right) - \frac{b}{a} \alpha + \beta \\
&= \frac{b}{a} \left( p^{u(M+1)} - 1 - \sum_{i \in \mathbb{I}_1} (p^{ui} - 1) - \sum_{i \in \mathbb{I}_2} 2(p^{ui} - 1) \right) \\
&\quad + \frac{b}{a} (1 - t - (M - K(p-1)(d + \varepsilon)) - 2K(p-1)(d + \varepsilon) - \alpha) + \beta \\
&= \frac{b}{a} \left( p^{u(M+1)} - 1 - \sum_{i \in \mathbb{I}_1} (p^{ui} - 1) - \sum_{i \in \mathbb{I}_2} 2(p^{ui} - 1) \right) \\
&\quad + \beta - \frac{b}{a} (c - t - \alpha - \rho - K(p-1)(a + 1 + \varepsilon)).
\end{aligned}$$

Thus

$$\begin{aligned}
bn + \beta &\equiv -mb \left( p^{u(M+1)} - 1 - \sum_{i \in \mathbb{I}_1} (p^{ui} - 1) - \sum_{i \in \mathbb{I}_2} 2(p^{ui} - 1) \right) \\
&\quad + \beta - mb (K(p-1)(a + 1 + \varepsilon) - (c - t - \alpha - \rho)) \\
&\equiv \beta - (\beta - 1) \equiv 1 \pmod{p}.
\end{aligned}$$

This shows that  $b_0 = 1$ . Given  $s$ , for  $j = 1, 2$  let  $I_{s,j}$  be the number of  $i \in \mathbb{I}_j$  with  $i \geq s$ . For  $s \in \mathbb{I}_j$  we have

$$\begin{aligned}
\lfloor \frac{bn + \beta}{p^{us}} \rfloor &= \lfloor \frac{b}{a} \left( p^{u(M+1-s)} - 1 - \sum_{s \leq i \in \mathbb{I}_1} (p^{u(i-s)} - 1) - \sum_{s \leq i \in \mathbb{I}_2} (2p^{u(i-s)} - 2) \right. \right. \\
&\quad \left. \left. - \sum_{s > i \in \mathbb{I}_1} \frac{1}{p^{u(s-i)}} - \sum_{s > i \in \mathbb{I}_2} \frac{2}{p^{u(s-i)}} - \frac{t}{p^{us}} - I_{s,1} - 2I_{s,2} + 1 \right) + \frac{\beta}{p^{us}} \right\rfloor \\
&= \frac{b}{a} \left( p^{u(M+1-s)} - 1 - \sum_{s \leq i \in \mathbb{I}_1} (p^{u(s-i)} - 1) - \sum_{s \leq i \in \mathbb{I}_2} (2p^{u(s-i)} - 2) \right) \\
&\quad + \lfloor \frac{b}{a} \left( - \sum_{s > i \in \mathbb{I}_1} \frac{1}{p^{u(s-i)}} - \sum_{s > i \in \mathbb{I}_2} \frac{2}{p^{u(s-i)}} - \frac{t}{p^{us}} - I_{s,1} - 2I_{s,2} + 1 \right) + \frac{\beta}{p^{us}} \rfloor \\
&\equiv -mb(-1 + I_{s,1} + 2I_{s,2} - j) - \lfloor \frac{b}{a}(-1 + I_{s,1} + 2I_{s,2}) \rfloor - 1 \pmod{p}.
\end{aligned}$$

Let  $I_{s,1} + I_{s,2} - 1 = cq_s + r_s$ , where  $0 \leq r_s < c$ . Then for  $s \in \mathbb{I}_j$  we have

$$b_{su} = mb(j - r_s) - \lfloor \frac{br_s}{a} \rfloor - 1 \equiv m(a \lfloor \frac{br_s}{a} \rfloor + a - b(r_s - j)) \pmod{p}.$$

Let us evaluate  $r_s$  for  $s \in \mathbb{I}_j$ . If  $j = 2$  then  $s = M - k(c - 1)$  for some  $k = 0, 1, \dots, K(p - 1)(d + \varepsilon) - 1$ . Thus

$$I_{s,1} = M - (M - k(c - 1)) + 1 - (k + 1), \quad I_{s,2} = k + 1.$$

So

$$cq_s + r_s = k(c - 1) + 1 + (k + 1) - 1 \equiv 1 \pmod{c}.$$

Hence  $r_s = 1$ , whenever  $s \in \mathbb{I}_2$ . Note that we have  $K(p - 1)(d + \varepsilon)$  times occurrence of  $r_s = 1$ .

Moreover, if  $j = 1$  then  $s = M - k(c - 1) - s'$  for some  $k = 0, 1, \dots, K(p - 1)(d + \varepsilon) - 1$  and  $s' = 1, 2, \dots, c - 2$ . Thus

$$I_{s,1} = M - (M - k(c - 1) - s') + 1 - (k + 1), \quad I_{s,2} = k + 1.$$

So

$$cq_s + r_s = k(c - 1) + s' + 1 + (k + 1) - 1 \equiv s' + 1 \pmod{c}.$$

Hence  $r_s = 2, \dots, c - 1$ , whenever  $s \in \mathbb{I}_1$ . Note that we have  $K(p - 1)(d + \varepsilon - 1)$  times occurrence of  $r_s = \rho + 2 - \varepsilon(c - 1), \dots, c - 1$  and  $K(p - 1)(d + \varepsilon)$  times occurrence of  $r_s = 2, \dots, \rho + 1 - \varepsilon(c - 1)$ .

Now we show that if  $s \in \mathbb{I}_1$  then  $b_{su} + 1 \not\equiv 0 \pmod{p}$  and if  $s \in \mathbb{I}_2$  then  $b_{su} + 1, b_{su} + 2 \not\equiv 0 \pmod{p}$ .

Let  $s \in \mathbb{I}_j$ . Then

$$b_{su} + 1 \equiv m(a \lfloor \frac{br_s}{a} \rfloor - b(r_s - j)) \pmod{p}.$$

Now if  $p \mid b_{su} + 1$  then  $p \mid a \lfloor \frac{br_s}{a} \rfloor - b(r_s - j)$ . The latter holds if and only if  $a \lfloor \frac{br_s}{a} \rfloor - b(r_s - j) = 0$ , since

$$|b(r_s - j) - a \lfloor \frac{br_s}{a} \rfloor| \leq a(\frac{br_s}{a} - \lfloor \frac{br_s}{a} \rfloor) + jb \leq a + 2b < p$$

Thus we should have  $a \mid b(r_s - j)$  which implies that  $c \mid r_s - j$ . This is a contradiction, since  $r_s < c$  and  $r_s \neq j$  whenever  $s \in \mathbb{I}_j$ .

Let  $s \in \mathbb{I}_2$ . Then

$$b_{su} + 2 \equiv m(a \lfloor \frac{br_s}{a} \rfloor - a - b(r_s - 2)) \pmod{p}.$$

We know that  $r_s = 1$  whenever  $s \in \mathbb{I}_2$ . Thus if  $p \mid b_{su} + 2$  then we should have  $p \mid a - b$ . The latter is impossible since  $p > a - b$ .

We therefore have

$$\begin{aligned}
\binom{an + \alpha}{bn + \beta} &\equiv \binom{p-1-t}{b_0} \prod_{s \in \mathbb{I}_1} (b_{su} + 1) \prod_{s \in \mathbb{I}_2} (b_{su} + 1)(b_{su} + 2) \\
&\equiv -(1+t) \prod_{r_s=2}^{\rho+1-\varepsilon(c-1)} (b_{su} + 1)^{K(p-1)(d+\varepsilon)} \prod_{r_s=\rho+2-\varepsilon(c-1)}^{c-1} (b_{su} + 1)^{K(p-1)(d+\varepsilon-1)} \\
&\quad \cdot (mb)^{K(p-1)(d+\varepsilon)} (m(b-a))^{K(p-1)(d+\varepsilon)} \\
&\equiv -(1+t) \\
&\equiv -(1+(p-1)-r) \\
&\equiv r \pmod{p}.
\end{aligned}$$

Note that there are infinitely many such  $n$ , since  $N$  was arbitrary.  $\square$

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